THE MOVING TRIHEDRON*

BY

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1. **Introduction.** A classical method of studying the metric differential geometry of curves and surfaces in three-dimensional space is based upon the use of a moving trihedron. A trihedron of reference is associated with an ordinary point of the curve, or surface, under consideration, and then the point is allowed to vary over the whole, or a suitably restricted portion, thereof. The theory which thus originates is particularly powerful in solving problems concerning two curves, or two surfaces, whose points are in one-to-one correspondence.

The theory of the moving trihedron in the study of curves, as outlined in §2 below, is due to Professor G. A. Bliss, who employed it effectively in his lectures on metric differential geometry at the University of Chicago. It was later also used by the author, to whom the extension to surfaces in the third and fourth sections is due. The essentially new feature of the treatment both for curves and for surfaces is found in the *recursion formulas* upon which the discussion rests. As these do not seem to have appeared elsewhere in the literature, the following exposition is designed to exhibit them and deduce some of their consequences.

2. **Curves.** The method of the moving trihedron as employed in the theory of curves will now be explained. Let us first of all establish an orthogonal cartesian coordinate system, which will be designated hereinafter as the *fixed* coordinate system. Referred to this system let the parametric equations of a real proper non-rectilinear analytic curve C be

$$x = x(s), \quad y = y(s), \quad z = z(s), \tag{1}$$

the parameter s being the arc length measured from some fixed point to the ordinary point $P(x, y, z)$ of C. Further, let us consider a point Q whose coordinates X, Y, Z are given as functions of s by equations of the form

$$X = X(s), \quad Y = Y(s), \quad Z = Z(s). \tag{2}$$

If these three functions of s are all constant, the point Q is fixed, relative to the fixed coordinate system, when the point P varies on the curve C. This case will be excluded hereinafter, unless the contrary is indicated. Then as s varies, the point P moves along the curve C, and the point Q traces a curve

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C_1 represented by the parametric equations (2). The points P, Q of the curves C, C_1 are in one-to-one correspondence, corresponding points being those associated with the same value of the parameter s.

At a point P of a curve C there is *the local coordinate system* with its origin at P, with the ξ-axis along the tangent, the η-axis along the principal normal, and the ζ-axis along the binormal. The equations of transformation between the coordinates X, Y, Z of the point Q that corresponds to P and the local coordinates ξ, η, ζ of Q are

$$\begin{aligned} X &= x + \alpha\xi + l\eta + \lambda\zeta, \\ Y &= y + \beta\xi + m\eta + \mu\zeta, \\ Z &= z + \gamma\xi + n\eta + \nu\zeta, \end{aligned} \tag{3}$$

wherein α, β, γ are the direction cosines, in the fixed coordinate system, of the tangent; l, m, n are those of the principal normal; and λ, μ, ν those of the binormal, of the curve C at the point P.

When the point P moves along the curve C, the local trihedron of C at P also moves, of course, and hence is appropriately called *the moving trihedron of the curve* C. The local coordinate system associated with the moving trihedron will be designated hereinafter as the *moving* coordinate system. The local coordinates ξ, η, ζ of the point Q corresponding to P are themselves functions of s. If these functions are constants, the point Q is rigidly attached to the moving trihedron, so that the motion of Q relative to the moving trihedron is zero.

For the purpose of investigating the relations of the curves C, C_1, it is convenient to know the direction cosines of the tangent, principal normal, and binormal of C_1 referred to the moving trihedron of C. In order to calculate these, some analytical consequences of equations (3) will next be deduced. If equations (3) are differentiated with respect to s, the results can be reduced, by means of the well known Frenet formulas, to

$$\begin{aligned} X' &= \alpha A_1 + lB_1 + \lambda C_1, \\ Y' &= \beta A_1 + mB_1 + \mu C_1, \\ Z' &= \gamma A_1 + nB_1 + \nu C_1 \qquad (X' = dX/ds, \cdots), \end{aligned} \tag{4}$$

wherein the coefficients A_1, B_1, C_1 are defined by the formulas

$$A_1 = 1 - \frac{\eta}{\rho} + \xi', \quad B_1 = \frac{\xi}{\rho} + \frac{\zeta}{\tau} + \eta', \quad C_1 = -\frac{\eta}{\tau} + \zeta', \tag{5}$$

and $1/\rho$, $1/\tau$ are respectively the curvature and torsion of the curve C at the point P. A second differentiation and reduction by the Frenet formulas lead to

$$X'' = \alpha A_2 + lB_2 + \lambda C_2 \tag{6}$$

and similar formulas for Y'', Z'', in which A_2, B_2, C_2 are defined by

$$A_2 = -\frac{B_1}{\rho} + A_1', \quad B_2 = \frac{A}{\rho} + \frac{C_1}{\tau} + B_1', \quad C_2 = -\frac{B_1}{\tau} + C_1'. \tag{7}$$

Repetition of the process gives

$$X''' = \alpha A_3 + lB_3 + \lambda C_3 \tag{8}$$

and similar formulas for Y''', Z''', in which A_3, B_3, C_3 are defined by

$$A_3 = -\frac{B_2}{\rho} + A_2', \quad B_3 = \frac{A_2}{\rho} + \frac{C_2}{\tau} + B_2', \quad C_3 = -\frac{B_2}{\tau} + C_2'. \tag{9}$$

An easy induction would yield

$$X^{(n)} = \alpha A_n + lB_n + \lambda C_n \tag{10}$$

and similar formulas for $Y^{(n)}$, $Z^{(n)}$, in which the coefficients A_n, B_n, C_n are given by the *recursion formulas*

$$A_n = -\frac{B_{n-1}}{\rho} + A_{n-1}', \quad B_n = \frac{A_{n-1}}{\rho} + \frac{C_{n-1}}{\tau} + B_{n-1}', \quad C_n = -\frac{B_{n-1}}{\tau} + C_{n-1}'. \tag{11}$$

It should be observed that A_n, B_n, C_n are the components in the moving coordinate system of that vector whose components in the fixed coordinate system are the derivatives $X^{(n)}$, $Y^{(n)}$, $Z^{(n)}$. Such a vector may be called a *derivative vector*. The components A_n, B_n, C_n are not themselves actually derivatives, but they behave in some respects like derivatives.

Some additional formulas will now be established. Let us make the convention that the arc length s_1 of the curve C_1, measured from some fixed point thereon, shall be an increasing function of the arc length s of C. Then squaring and adding equations (4), and taking the positive square root, we find

$$\frac{ds_1}{ds} = (\sum X'^2)^{1/2} = (\sum A_1^2)^{1/2}, \tag{12}$$

the summation being for cyclical permutations. Easy calculations now yield

$$\begin{aligned} \frac{ds}{ds_1} &= \frac{1}{(\sum A_1^2)^{1/2}}, \\ \frac{d^2 s}{ds_1^2} &= -\frac{\sum A_1 A_2}{(\sum A_1^2)^2}. \end{aligned} \tag{13}$$

Formulas for higher derivatives of s with respect to s_1 could be calculated but will not be needed in what is to follow.

Elementary calculus supplies the formulas

$$\frac{dX}{ds_1} = X' \frac{ds}{ds_1},$$

$$\frac{d^2X}{ds_1^2} = X''\left(\frac{ds}{ds_1}\right)^2 + X'\frac{d^2s}{ds_1^2}, \tag{14}$$

$$\frac{d^3X}{ds_1^3} = X'''\left(\frac{ds}{ds_1}\right)^3 + 3X''\frac{ds}{ds_1}\frac{d^2s}{ds_1^2} + X'\frac{d^3s}{ds_1^3},$$

and similar ones for the derivatives of Y, Z. The second of (14) can be reduced to

$$\frac{d^2X}{ds_1^2} = \alpha L + lM + \lambda N, \tag{15}$$

where L, M, N are defined by

$$L = \frac{1}{(\sum A_1^2)^2}(A_2 \sum A_1^2 - A_1 \sum A_1 A_2),$$

$$M = \frac{1}{(\sum A_1^2)^2}(B_2 \sum A_1^2 - B_1 \sum A_1 A_2), \tag{16}$$

$$N = \frac{1}{(\sum A_1^2)^2}(C_2 \sum A_1^2 - C_1 \sum A_1 A_2).$$

Direct calculation results in

$$\frac{dY}{ds_1}\frac{d^2Z}{ds_1^2} - \frac{d^2Y}{ds_1^2}\frac{dZ}{ds_1} = \left(\frac{ds}{ds_1}\right)^3(\alpha P + lQ + \lambda R), \tag{17}$$

where P, Q, R are defined by

$$P = B_1C_2 - B_2C_1, \qquad Q = C_1A_2 - C_2A_1, \qquad R = A_1B_2 - A_2B_1. \tag{18}$$

Finally, the curvature $1/\rho_1$ and the torsion $1/\tau_1$ at a point of the curve C_1 can without difficulty be shown to be given by the formulas

$$\frac{1}{\rho_1^2} = \sum L^2 = \frac{\sum P^2}{(\sum A_1^2)^3},$$

$$\frac{1}{\tau_1} = -\frac{1}{\sum P^2}\begin{vmatrix} A_1 & B_1 & C_1 \\ A_2 & B_2 & C_2 \\ A_3 & B_3 & C_3 \end{vmatrix}. \tag{19}$$

The direction cosines of the tangent, principal normal, and binormal at a point Q of the curve C_1, referred to the moving trihedron of the curve C at the corresponding point P, can now be found by the familiar equations of transformation of direction cosines. For example, *the direction cosines of the tangent* of C_1, referred to the fixed coordinate system, are known to be

$$\frac{dX}{ds_1}, \quad \frac{dY}{ds_1}, \quad \frac{dZ}{ds_1}. \tag{20}$$

Therefore, by equations (4) and the first of (14), the direction cosines of the tangent referred to the moving trihedron are found to be

$$A_1\frac{ds}{ds_1}, \quad B_1\frac{ds}{ds_1}, \quad C_1\frac{ds}{ds_1}. \tag{21}$$

Similarly, *the direction cosines of the principal normal* of C_1 in the fixed coordinate system are known to be

$$\rho_1\frac{d^2X}{ds_1^2}, \quad \rho_1\frac{d^2Y}{ds_1^2}, \quad \rho_1\frac{d^2Z}{ds_1^2}, \tag{22}$$

and in the moving coordinate system are found to be

$$\rho_1 L, \quad \rho_1 M, \quad \rho_1 N. \tag{23}$$

Finally, *the direction cosines of the binormal* of C_1 in the fixed coordinate system are known to be

$$\rho_1\left(\frac{dY}{ds_1}\frac{d^2Z}{ds_1^2} - \frac{d^2Y}{ds_1^2}\frac{dZ}{ds_1}\right) \tag{24}$$

and two similar expressions; hence these direction cosines in the moving coordinate system are

$$\rho_1\left(\frac{ds}{ds_1}\right)^3 P, \quad \rho_1\left(\frac{ds}{ds_1}\right)^3 Q, \quad \rho_1\left(\frac{ds}{ds_1}\right)^3 R. \tag{25}$$

The direction cosines of the tangent, principal normal, and binormal of the curve C_1, referred to the moving trihedron of the curve C, are therefore respectively proportional to

$$A_1, B_1, C_1;\ L, M, N;\ P, Q, R. \tag{26}$$

The general theory just outlined is capable of extensive applications. It forms a powerful tool for the study of curves which are transforms of a given curve, such as involutes, evolutes, parallel curves, and so on. But limitations of space do not permit inclusion of such developments here.

3. **Surfaces.** First of all, some preliminary formulas in surface theory will be collected for subsequent use. Let us consider a real proper analytic surface S, not a sphere or a plane, whose parametric equations in a fixed coordinate system are

$$x = x(u, v), \quad y = y(u, v), \quad z = z(u, v). \tag{27}$$

Let the lines of curvature be the parametric curves on the surface S, so that

$$F = 0, \quad D' = 0, \tag{28}$$

in the classical notation of Eisenhart and Bianchi. The direction cosines α^u, β^u, γ^u of the u-tangent at a point $P(x, y, z)$ of S are given by the formulas

$$\alpha^u = \frac{x_u}{E^{1/2}}, \quad \beta^u = \frac{y_u}{E^{1/2}}, \quad \gamma^u = \frac{z_u}{E^{1/2}}. \tag{29}$$

Similarly, the direction cosines α^v, β^v, γ^v of the v-tangent of S at P are given by

$$\alpha^v = \frac{x_v}{G^{1/2}}, \quad \beta^v = \frac{y_v}{G^{1/2}}, \quad \gamma^v = \frac{z_v}{G^{1/2}}, \tag{30}$$

and the direction cosines a, b, c of the normal of S at P by

$$a = \frac{y_u z_v - y_v z_u}{(EG)^{1/2}}, \quad b = \frac{z_u x_v - z_v x_u}{(EG)^{1/2}}, \quad c = \frac{x_u y_v - x_v y_u}{(EG)^{1/2}}. \tag{31}$$

The curvilinear parametric equations of any curve C through the point P on the surface S are

$$u = u(s), \quad v = v(s), \tag{32}$$

the parameter s being the arc length measured from some fixed point of C. The direction cosines α, β, γ of the tangent of C at P are expressed by the formulas

$$\alpha = x_u u' + x_v v', \quad \beta = y_u u' + y_v v', \quad \gamma = z_u u' + z_v v' \quad (u' = du/ds, \cdots). \tag{33}$$

Let θ be the angle from the positive half of the u-tangent to the positive half of the tangent of the curve C at the point P. Then one has

$$\cos\theta = E^{1/2}u', \quad \sin\theta = G^{1/2}v'. \tag{34}$$

The principal normal curvatures $1/R_1$, $1/R_2$ of the surface S are given by the formulas

$$\frac{1}{R_1} = \frac{D}{E}, \quad \frac{1}{R_2} = \frac{D''}{G}, \tag{35}$$

and the geodesic curvatures $1/r_1$, $1/r_2$ of the lines of curvature by

$$\frac{1}{r_1} = -\frac{E_v}{2EG^{1/2}}, \quad \frac{1}{r_2} = +\frac{G_u}{2GE^{1/2}}, \tag{36}$$

the subscript 1 in each case denoting the function associated with the u-curve, and 2 that with the v-curve, at a point P.

Formulas analogous to the Frenet formulas can be established for the local trihedron whose edges are the tangents of the lines of curvature and the normal at a point P of a surface S. These formulas express the derivatives, with respect to the arc length s of a curve C, of the direction cosines of the three edges of the local trihedron linearly in terms of these cosines themselves, the coefficients depending upon the functions θ, R_1, R_2, r_1, r_2. In fact, actual calculation, the details of which will be omitted, leads to the formulas in question, namely,

$$\begin{aligned} (\alpha^u)' &= \left(\frac{\cos\theta}{r_1} + \frac{\sin\theta}{r_2}\right)\alpha^v + \frac{\cos\theta}{R_1}a, \\ (\alpha^v)' &= -\left(\frac{\cos\theta}{r_1} + \frac{\sin\theta}{r_2}\right)\alpha^u + \frac{\sin\theta}{R_2}a, \\ a' &= -\frac{\cos\theta}{R_1}\alpha^u - \frac{\sin\theta}{R_2}\alpha^v \qquad (a' = da/ds, \cdots). \end{aligned} \tag{37}$$

and similar formulas for the remaining derivatives. With these should be associated the easily verified result

$$x' = \cos\theta\,\alpha^u + \sin\theta\,\alpha^v, \tag{38}$$

with similar expressions for y', z'.

Let us establish a local coordinate system at a point P of a surface S, referred to its lines of curvature, with the origin at P, the ξ-axis along the u-tangent, the η-axis along the v-tangent, and the ζ-axis along the normal of S at P. The equations of transformation between the coordinates X, Y, Z of any point Q (supposed to be functions of u, v, and referred to the fixed coordinate system) and the local coordinates ξ, η, ζ of Q are

$$\begin{aligned} X &= x + \alpha^u\xi + \alpha^v\eta + a\zeta, \\ Y &= y + \beta^u\xi + \beta^v\eta + b\zeta, \\ Z &= z + \gamma^u\xi + \gamma^v\eta + c\zeta. \end{aligned} \tag{39}$$

Recursion formulas exactly analogous to those in §1 can be obtained by repeated differentiation of these equations. Differentiating once with respect to the arc length s of the curve C we find, by means of (37), (38),

$$X' = \alpha^u A_1 + \alpha^v B_1 + a C_1 \tag{40}$$

and similar formulas for Y', Z', in which the coefficients A_1, B_1, C_1 are defined by

$$\begin{aligned} A_1 &= \cos\theta\left(1 - \frac{\eta}{r_1} - \frac{\zeta}{R_1}\right) - \sin\theta\frac{\eta}{r_2} + \xi', \\ B_1 &= \cos\theta\frac{\xi}{r_1} + \sin\theta\left(1 + \frac{\xi}{r_2} - \frac{\zeta}{R_2}\right) + \eta', \\ C_1 &= \cos\theta\frac{\xi}{R_1} + \sin\theta\frac{\eta}{R_2} + \zeta'. \end{aligned} \tag{41}$$

A second differentiation, followed by appropriate reduction, gives

$$X'' = \alpha^u A_2 + \alpha^v B_2 + a C_2, \tag{42}$$

where A_2, B_2, C_2 are defined by

$$\begin{aligned} A_2 &= \cos\theta\left(-\frac{B_1}{r_1} - \frac{C_1}{R_1}\right) - \sin\theta\frac{B_1}{r_2} + A_1', \\ B_2 &= \cos\theta\frac{A_1}{r_1} + \sin\theta\left(\frac{A_1}{r_2} - \frac{C_1}{R_2}\right) + B_1', \\ C_2 &= \cos\theta\frac{A_1}{R_1} + \sin\theta\frac{B_1}{R_2} + C_1'. \end{aligned} \tag{43}$$

In general we find

$$X^{(n)} = \alpha^u A_n + \alpha^v B_n + a C_n \tag{44}$$

where the local components A_n, B_n, C_n of the derivative vector $X^{(n)}$, $Y^{(n)}$, $Z^{(n)}$ are given by the *recursion formulas*

$$\begin{aligned} A_n &= \cos\theta\left(-\frac{B_{n-1}}{r_1} - \frac{C_{n-1}}{R_1}\right) - \sin\theta\frac{B_{n-1}}{r_2} + A_{n-1}', \\ B_n &= \cos\theta\frac{A_{n-1}}{r_1} + \sin\theta\left(\frac{A_{n-1}}{r_2} - \frac{C_{n-1}}{R_2}\right) + B_{n-1}', \\ C_n &= \cos\theta\frac{A_{n-1}}{R_1} + \sin\theta\frac{B_{n-1}}{R_2} + C_{n-1}'. \end{aligned} \tag{45}$$

With the definitions of the functions A_n, B_n, C_n employed in this section, the formulas (12), $\cdots$, (26) of §2 can easily be shown to be equally valid for the local trihedron of surface theory. One thus obtains a theory differing from that of §2 only in two particulars; namely, the curve C is now supposed to lie on a given surface; and a different local trihedron is now being associated with the curve C. These considerations will not be pursued further here.

The principal interest in the theory of the moving trihedron in surface theory arises when the point P, instead of tracing a curve C on the surface S, is allowed to vary over a suitably restricted region of S. In this case the local components of the partial derivative vectors are required. These may be obtained by specializing equations (37), (38), and (40), $\cdots$, (45), if it is kept in mind that

$$ds^u = E^{1/2}du, \quad ds^v = G^{1/2}dv, \tag{46}$$

where s^u, s^v denote arc lengths on the parametric curves. The required formulas can also be calculated directly. Either way one finds

$$X_u = A^u\alpha^u + B^u\alpha^v + C^u a, \quad X_v = A^v\alpha^u + B^v\alpha^v + C^v a, \tag{47}$$

and similar formulas for the first partial derivatives of Y, Z, where the coefficients $A^u, \cdots, A^v, \cdots$ are defined by the formulas

$$\begin{aligned}
&\frac{A^u}{E^{1/2}} = 1 - \frac{\eta}{r_1} - \frac{\zeta}{R_1} + \frac{\xi_u}{E^{1/2}}, && \frac{A^v}{G^{1/2}} = -\frac{\eta}{r_2} + \frac{\xi_v}{G^{1/2}},\\
&\frac{B^u}{E^{1/2}} = \frac{\xi}{r_1} + \frac{\eta_u}{E^{1/2}}, && \frac{B^v}{G^{1/2}} = 1 + \frac{\xi}{r_2} - \frac{\zeta}{R_2} + \frac{\eta_v}{G^{1/2}},\\
&\frac{C^u}{E^{1/2}} = \frac{\xi}{R_1} + \frac{\zeta_u}{E^{1/2}}, && \frac{C^v}{G^{1/2}} = \frac{\eta}{R_2} + \frac{\zeta_v}{G^{1/2}}.
\end{aligned} \tag{48}$$

Further differentiation, followed by appropriate reductions, yields

$$\begin{aligned}
X_{uu} &= A^{uu}\alpha^u + B^{uu}\alpha^v + C^{uu}a,\\
X_{uv} &= A^{uv}\alpha^u + B^{uv}\alpha^v + C^{uv}a,\\
X_{vv} &= A^{vv}\alpha^u + B^{vv}\alpha^v + C^{vv}a,
\end{aligned} \tag{49}$$

and similar formulas for the second partial derivatives of Y, Z, where

$$\begin{aligned}
\frac{A^{uu}}{E^{1/2}} &= -\frac{B^u}{r_1} - \frac{C^u}{R_1} + \frac{A_u{}^u}{E^{1/2}},\\
\frac{B^{uu}}{E^{1/2}} &= \frac{A^u}{r_1} + \frac{B_u{}^u}{E^{1/2}},\\
\frac{C^{uu}}{E^{1/2}} &= \frac{A^u}{R_1} + \frac{C_u{}^u}{E^{1/2}},
\end{aligned} \tag{50}$$

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(50)\qquad
\begin{aligned}
\frac{A^{uv}}{G^{1/2}} &= -\frac{B^u}{r_2}+\frac{A_v}{G^{1/2}}, &\qquad \frac{A^{uv}}{E^{1/2}} &= -\frac{B^v}{r_1}-\frac{C^v}{R_1}+\frac{A^v_u}{E^{1/2}},\\
\frac{B^{uv}}{G^{1/2}} &= \frac{A^u}{r_2}-\frac{C^u}{R_2}+\frac{B^u_v}{G^{1/2}}, & \frac{B^{uv}}{E^{1/2}} &= \frac{A^v}{r_1}+\frac{B^v_u}{E^{1/2}},\\
\frac{C^{uv}}{G^{1/2}} &= \frac{B^u}{R_2}+\frac{C^u_v}{G^{1/2}}, & \frac{C^{uv}}{E^{1/2}} &= \frac{B^v}{R_1}+\frac{C^v_u}{E^{1/2}},\\
\frac{A^{vv}}{G^{1/2}} &= -\frac{B^v}{r_2}+\frac{A^v_v}{G^{1/2}}, & &\\
\frac{B^{vv}}{G^{1/2}} &= \frac{A^v}{r_2}-\frac{C^v}{R_2}+\frac{B^v_v}{G^{1/2}}, & &\\
\frac{C^{vv}}{G^{1/2}} &= \frac{B^v}{R_2}+\frac{C^v_v}{G^{1/2}}\cdot & &
\end{aligned}
$$

The calculation of the local components of derivative vectors of higher order than the second is now purely mechanical, but none of them will be used hereinafter, and recursion formulas for the local components of the derivative vectors of the nth order need not be written.

Let us suppose for the present that the locus of the point Q, when u, v vary, is a proper surface S_1, and let the six fundamental coefficients and other functions for this surface be indicated by subscripts 1. For *the first three fundamental coefficients* we find, by easy calculations from equations (47),

$$(51)\qquad E_1 = \sum A^{u^2}, \quad F_1 = \sum A^u A^v, \quad G_1 = \sum A^{v^2},$$

whence

$$(52)\qquad H_1^2 = E_1G_1 - F_1^2 = \sum (B^uC^v - B^vC^u)^2.$$

The direction cosines of the u-tangent at a point of the surface S_1, referred to the moving trihedron of the surface S at the corresponding point P, are found to be

$$(53)\qquad \frac{A^u}{E_1^{1/2}}, \quad \frac{B^u}{E_1^{1/2}}, \quad \frac{C^u}{E_1^{1/2}},$$

and similarly *the direction cosines of the v-tangent* are

$$(54)\qquad \frac{A^v}{G_1^{1/2}}, \quad \frac{B^v}{G_1^{1/2}}, \quad \frac{C^v}{G_1^{1/2}},$$

while *the direction cosines of the normal* of S_1 are

$$(55) \qquad \frac{1}{H_1}(B^uC^v - B^vC^u), \quad \frac{1}{H_1}(C^uA^v - C^vA^u), \quad \frac{1}{H_1}(A^uB^v - A^vB^u).$$

Finally, *the second fundamental coefficients* for the surface S_1 are found to be given by

$$(56) \qquad \begin{aligned} D_1 &= \frac{1}{H_1}\sum A^{uu}(B^uC^v - B^vC^u), \\ D_1' &= \frac{1}{H_1}\sum A^{uv}(B^uC^v - B^vC^u), \\ D_1'' &= \frac{1}{H_1}\sum A^{vv}(B^uC^v - B^vC^u). \end{aligned}$$

Since the six fundamental coefficients for the surface S_1 have been calculated, it is only a formal matter to write the expressions for the mean and total curvatures, the equation of the lines of curvature, etc., for the surface S_1 in terms of the components $A^u, \cdots, A^{vv}, \cdots$.

4. **Applications.** Some applications of the theory of the moving trihedron in the theory of surfaces, as explained in the preceding section, will now engage our attention. First of all, equations (47) and the similar equations for Y, Z show that the point $Q(X, Y, Z)$ is fixed relative to the fixed coordinate system if, and only if,

$$A^u = B^u = C^u = A^v = B^v = C^v = 0.$$

Equations (48) now yield *necessary and sufficient conditions that the point* Q *be fixed relative to the fixed coordinate system*, namely,

$$(57) \qquad \begin{aligned} \xi_u &= E^{1/2}\left(-1 + \frac{\eta}{r_1} + \frac{\zeta}{R_1}\right), & \xi_v &= G^{1/2}\left(\frac{\eta}{r_2}\right), \\ \eta_u &= E^{1/2}\left(-\frac{\xi}{r_1}\right), & \eta_v &= G^{1/2}\left(-1 - \frac{\xi}{r_2} + \frac{\zeta}{R_2}\right), \\ \zeta_u &= E^{1/2}\left(-\frac{\xi}{R_1}\right), & \zeta_v &= G^{1/2}\left(-\frac{\eta}{R_2}\right). \end{aligned}$$

These conditions are very useful in solving *envelope problems* of a type which will now be described. Let us consider a surface S referred to its lines of curvature, and a two-parameter family of surfaces such that one of them, S_1, is associated with each point P of S. Let the equation of S_1 be

$$f(\xi, \eta, \zeta, u, v) = 0,$$

in which ξ, η, ζ are local coordinates referred to the moving trihedron of S

at P, and u, v are the curvilinear coordinates of P. It may be required to find the envelope of the surface S_1 when the point P describes a curve or a region of the surface S. The usual method of investigating the envelope entails the differentiation of the functions ξ, η, ζ with respect to u and v, and it will next be shown that the conditions (57) are precisely the needed *formulas for the differentiation of local point coordinates*. For this purpose, let us observe that the result of solving equations (39) for ξ, η, ζ is

$$\begin{aligned} \xi &= \alpha^u(X - x) + \beta^u(Y - y) + \gamma^u(Z - z), \\ \eta &= \alpha^v(X - x) + \beta^v(Y - y) + \gamma^v(Z - z), \\ \zeta &= a(X - x) + b(Y - y) + c(Z - z). \end{aligned} \tag{58}$$

Consequently the equation of the surface S_1 referred to the *fixed* coordinate system can be written in the form

$$f\left(\sum \alpha^u(X - x), \quad \sum \alpha^v(X - x), \quad \sum a(X - x), u, v\right) = 0,$$

the summation being for cyclical permutations. Since u, v occur explicitly and also in $\alpha^u, \alpha^v, a, x, \cdots$, but not in X, Y, Z, partial differentiation yields

$$\begin{aligned} f_\xi \xi_u + f_\eta \eta_u + f_\zeta \zeta_u + f_u &= 0, \\ f_\xi \xi_v + f_\eta \eta_v + f_\zeta \zeta_v + f_v &= 0, \end{aligned} \tag{59}$$

where the partial derivatives of ξ, η, ζ are to be calculated from equations (58) by direct differentiation with X, Y, Z fixed. If use is made of equations (37), suitably specialized, to obtain the partial derivatives of $\alpha^u, \alpha^v, a, \cdots$ as linear combinations of $\alpha^u, \alpha^v, a, \cdots$, and if equations (58) themselves are then employed to express the derivatives of ξ, η, ζ as functions of ξ, η, ζ, the result of the differentiation can be reduced to equations (57), as was to be shown.

By way of illustration let us consider the osculating plane of the u-curve at the point P of the surface S. If the equation of this plane, referred to the fixed coordinate system, is written in the usual form, the equations of transformation (39) and the equations (29) together with the equations obtained by differentiating the latter with respect to u can be used to show that the local equation of *the osculating plane of the u-curve* is

$$\frac{\eta}{R_1} - \frac{\zeta}{r_1} = 0. \tag{60}$$

If this equation is differentiated with respect to v, the result can be reduced by means of one of the conditions of Codazzi, namely,

$$(61)\qquad \left(\frac{1}{R_1}\right)_v = -\frac{G^{1/2}}{r_1}\left(\frac{1}{R_2}-\frac{1}{R_1}\right),$$

to

$$(62)\qquad \frac{\xi}{r_2}-\frac{\eta}{r_1}+1+\left[\frac{1}{G^{1/2}}\left(\frac{1}{r_1}\right)_v-\frac{1}{R_1R_2}\right]R_1\zeta = 0,$$

provided that the surface S is not developable. Equations (60), (62) taken together are the equations of the characteristic of the osculating plane of the u-curve when v varies. The equations of the orthogonal projection of this characteristic onto the tangent plane are

$$(63)\qquad \zeta = 0,\quad \frac{\xi}{r_2}-\frac{\eta}{r_1}+1+\left[\frac{1}{G^{1/2}}\left(\frac{1}{r_1}\right)_v-\frac{1}{R_1R_2}\right]r_1\eta = 0.$$

Since the equations of *the ray of the lines of curvature*, namely, the straight line joining the Laplace transformed points or ray-points $(0, r_1, 0)$ and $(-r_2, 0, 0)$, are

$$(64)\qquad \zeta = 0,\quad \frac{\xi}{r_2}-\frac{\eta}{r_1}+1 = 0,$$

it follows that *the orthogonal projection of the characteristic of the osculating plane of the u-curve, when v varies, onto the tangent plane coincides with the ray if, and only if,*

$$(65)\qquad \frac{1}{G^{1/2}}\left(\frac{1}{r_1}\right)_v-\frac{1}{R_1R_2} = 0.$$

Differentiation of equation (62) would enable us to find the edge of regression of the developable enveloped by the osculating plane of the u-curve when v varies.

The equation of *the rectifying plane of the u-curve* at the point P can easily be shown to be

$$(66)\qquad \frac{\eta}{r_1}+\frac{\zeta}{R_1} = 0,$$

since this plane must contain the tangent line, $\eta=\zeta=0$, and must be perpendicular to the osculating plane (60) of the u-curve. The equations of the characteristic of this plane when v varies can be found by the method just used for the osculating plane. The equations of the orthogonal projection of this line onto the tangent plane turn out to differ from equations (63) only in that the sign of η has been changed. Therefore *the projections onto the tangent plane of the characteristics of the osculating plane and rectifying plane of the* *u-curve, when v varies, are symmetrically placed with respect to the tangent line of the u-curve.*

The machinery of the local trihedron can be efficiently used to investigate the focal surfaces of the congruence of normals of a surface S, and the Laplace transformed nets of the lines of curvature on S, but as the principal results are well known, this study need not be entered upon here. It may be worthy of comment, however, that it is easy to locate the centers of the osculating circle and osculating sphere of the u-curve. Differentiating the equation $\xi=0$ of the normal plane of the u-curve with respect to u we obtain the equations of *the polar line* of the u-curve at the point P, namely,

$$\xi = 0, \quad \frac{\eta}{r_1} + \frac{\zeta}{R_1} = 1. \tag{67}$$

This line intersects the osculating plane (60) in the center of *the osculating circle* of the u-curve, whose coordinates are thus found to be

$$0, \qquad \frac{\rho_1^2}{r_1}, \qquad \frac{\rho_1^2}{R_1},$$

the radius of curvature ρ_1 of the u-curve being given by

$$\frac{1}{\rho_1^2} = \frac{1}{r_1^2} + \frac{1}{R_1^2}\,. \tag{68}$$

The polar line meets the surface normal, $\xi=\eta=0$, at the center of the principal normal curvature corresponding to the u-curve $(0, 0, R_1)$, and meets the v-tangent, $\xi=\zeta=0$, at the ray-point of the u-curve $(0, r_1, 0)$. A second differentiation with respect to u and solution of three simultaneous equations yield the coordinates of the center of *the osculating sphere* of the u-curve, namely,

$$0, \ \rho_1\left(\frac{\rho_1}{r_1} + \frac{\tau_1\rho_{1u}}{R_1E^{1/2}}\right), \quad \rho_1\left(\frac{\rho_1}{R_1} - \frac{\tau_1\rho_{1u}}{r_1E^{1/2}}\right),$$

the torsion $1/\tau_1$ of the u-curve being given by

$$E^{1/2}\frac{1}{\rho_1^2}\,\frac{1}{\tau_1} = \frac{1}{R_1}\left(\frac{1}{r_1}\right)_u - \frac{1}{r_1}\left(\frac{1}{R_1}\right)_u. \tag{69}$$

The usual formula for the radius of the osculating sphere, in terms of ρ_1, τ_1, and their derivatives with respect to the arc length of the u-curve, could easily be used to write down a condition necessary and sufficient that one family of lines of curvature, namely, the u-curves, on a surface be spherical. Similar results can be obtained with the u-curves and v-curves interchanged.

Necessary and sufficient conditions that the surface S_1 generated by the point Q may be obtainable from the surface S by a translation can be found in the following way. In case these surfaces differ only by a translation, the differences $X-x$, $Y-y$, $Z-z$ are constants. Differentiating equations (39) under this assumption we find the required conditions, namely,

$$
\begin{aligned}
A^u &= E^{1/2}, & A^v &= 0, \\
B^u &= 0, & B^v &= G^{1/2}, \\
C^u &= 0, & C^v &= 0.
\end{aligned}
\tag{70}
$$

These conditions are equivalent to the conditions (57) with the modification that the terms consisting of the number -1 must be deleted from the parentheses in the formulas for ξ_u, η_v therein.

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